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STABILITY OF TWO-FLUID WHEEL FLOWS WITH AN IMPOSED UNIFORM AXIAL MAGNETIC FIELD

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ABSTRACT

The stability of an incompressible two-fluid wheel flow to infinitesimal helical disturbances is studied. The inner fluid is heavy and has infinite electrical conductivity, while the outer fluid is light and is nonconducting. An axial magnetic field is externally imposed on both fluids. This configuration may be viewed as a Rayleigh-Taylor problem in the frame of the rotating fluid and is dynamically unstable. Growth rates increase with increasing axial wavelength and azimuthal mode numbers, but decrease with increasing axial magnetic field. By increasing the magnetic field sufficiently, the system can be made stable to short axial wavelength disturbances for any azimuthal mode.

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SUMMARY

The stability of an incompressible two-fluid wheel flow to infinitesimal helical disturbances is studied. The inner fluid is heavy and has infinite electrical conductivity, while the outer fluid is light and is nonconducting. An axial magnetic field is externally imposed on both fluids. This configuration may be viewed as a Rayleigh-Taylor problem in the frame of the rotating fluid and is dynamically unstable. Growth rates increase with increasing axial wavelength and azimuthal mode numbers, but decrease with increasing axial magnetic field. By increasing the magnetic field sufficiently, the system can be made stable to short axial wavelength disturbances for any azimuthal mode.

INTRODUCTION

Vortex containment of a heavy fissioning gas surrounded by a lighter coolant gas has been suggested for possible application to gaseous core nuclear rockets (refs. 1 to 3). Another application of the vortex containment principle is found in the vortex magnetohydrodynamic (MHD) generator (ref. 4). The problem investigated in this report was originally conceived to study the stability characteristics of the two-fluid wheel-flow reactor concept of Evvard (ref. 3). In this concept, a core of heavy fissioning gas in solid-body rotation is surrounded by a lighter coolant gas also in solid-body rotation at the same angular frequency. Since the centrifugal force is directed radially outward from the heavy inner fluid towards the lighter outer fluid, this flow configuration is dynamically unstable. When viewed from the rotating reference frame, the situation is analogous to the well-known Rayleigh-Taylor instability problem where a heavy fluid is supported against gravity by a lighter fluid. Reshotko and Monnin (ref. 5) investigated the nature of this two-fluid wheel-flow instability from hydrodynamic considerations only.

They found the flow always unstable with growth rates that increase with increasing axial, as well as azimuthal, wave number.

Because the gaseous core is highly ionized and therefore electrically conducting, Evvard suggested that an externally imposed axial magnetic field might be used to stabilize the flow. Obviously, any realistic evaluation of the stability problems involved in this complex flow system would have to be determined by experiment. But the proposed concept does suggest an idealized flow configuration whose stability can be investigated mathematically by a straightforward perturbation analysis.

This idealized flow problem, treated in the present report, consists of two inviscid immiscible fluids of constant and uniform density separated by a cylindrical interface. The inner fluid is a perfect electrical conductor, while the outer fluid has the electrical properties of a vacuum. There is an externally imposed axial magnetic field which, at equilibrium, is uniform and has the same value in both fluids. This is a hydromagnetic flow in which the electromagnetic fields are coupled to the hydrodynamic flow fields. The hydromagnetic equations, with scalar pressure, were used to describe the flow.

Although the correspondence between the real flow problem and the idealized one is tenuous indeed, the idealized problem possesses the advantage that it can be readily solved and some of the results should be of use in the design of an experiment and in the interpretation of results. The idealized problem is also of interest in its own right since it has not been previously investigated. Furthermore, in hydromagnetic fluids such as a liquid metal, the sharp boundary and the infinite electrical conductivity would correspond to a realistic physical situation.

Wilhelm (ref. 6) has investigated the stability of an idealized two-fluid wheel-flow configuration where the inner fluid was a perfect conductor and the outer fluid was a nonconducting gas. The equilibrium magnetic field was an azimuthal field produced by an axial sheet current at the interface between the inner and outer fluids. The basic equations used in the present report are identical to those of Wilhelm.

In the idealized model, viscous effects were omitted, but as Chandrasekhar (ref. 7) notes for rotating flows, viscosity severely diminishes the growth rate of short-wavelength disturbances. We should therefore expect the short-wavelength results of this report to be pessimistic in the predictions of stability. Finite resistivity effects have also been omitted in the analysis, but Kruskal and Schwarzschild (ref. 8) have found that for the kind of problem under consideration, namely large electrical conductivity, the infinite conductivity assumption preserves the essential features of the real situation.

In this report, the hydromagnetic stability problem is solved by the normal-mode technique. We begin by writing the continuity, momentum, Ohm's law, and Maxwell equations for the inner and outer fluids. These equations are linearized by performing a perturbation analysis about an assumed equilibrium flow. Solutions to these equations are obtained for the inner and outer fluids, and the interfacial boundary conditions are derived. Simultaneous solution of the volume and boundary equations yields the dispersion relation. The dispersion relation is solved for complex frequencies and real wave numbers. The results suggest some methods that might be employed to stabilize this type of flow. Dimensional analysis is applied to determine the degree to which the idealized model is approximated by the wheel-flow reactor plasma, and by the liquid metals sodium, potassium, and mercury. The relevance of the idealized model to situations where finite amplitude disturbances and finite electrical conductivity exist is discussed in a qualitative way.

BASIC FLOW EQUATIONS

Inner-Fluid Equations

The hydromagnetic equations, in MKS units, are used for the inner fluid:

$$\rho\left[\frac{\partial \underline{V}^*}{\partial t} + (\underline{V}^* \cdot \nabla)\underline{V}^*\right] = \underline{J}^* \times \underline{B}^* - \nabla P^* \tag{1}$$

$$\nabla \cdot \underline{V}^* = 0 \tag{2}$$

$$\underline{E}^* + \underline{V}^* \times \underline{B}^* = 0 \tag{3}$$

$$\nabla \times \underline{B}^* = \mu_o \underline{J}^* \tag{4}$$

$$\nabla \cdot \underline{B}^* = 0 \tag{5}$$

$$\nabla \times \underline{E}^* = -\frac{\partial \underline{B}^*}{\partial t} \tag{6}$$

The asterisks indicate that the quantities are dimensional quantities. (All symbols are defined in appendix A.) In these equations viscosity, electrical resistivity, space charge, and displacement current have all been set equal to zero.

The electric field $\underline{E}^*$ and current density $\underline{J}^*$ can be eliminated from equations (1) and (6) by use of equations (3) and (4) to yield (ref. 6)

$$\rho\left[\frac{\partial \underline{V}^*}{\partial t} + (\underline{V}^* \cdot \nabla)\underline{V}^*\right] = \frac{1}{\mu_o}(\underline{B}^* \cdot \nabla)\underline{B}^* - \nabla\left(P^* + \frac{B^{*2}}{2\mu_o}\right) \tag{7}$$

$$\nabla \cdot \underline{V}^* = 0 \tag{8}$$

$$\nabla \cdot \underline{B}^* = 0 \tag{9}$$

$$\nabla \times (\underline{V}^* \times \underline{B}^*) = \frac{\partial \underline{B}^*}{\partial t^*} \tag{10}$$

Outer-Fluid Equations

Since the outer fluid is an electrically nonconducting fluid, the electromagnetic equations are uncoupled from the hydrodynamic equations. Thus, for the outer fluid, the hydrodynamic equations are

$$\rho\left[\frac{\partial \underline{V}^*}{\partial t} + (\underline{V}^* \cdot \nabla)\underline{V}^*\right] = -\nabla P^* \tag{11}$$

$$\nabla \cdot V^* = 0 \tag{12}$$

The electromagnetic equations in the outer fluid are Maxwell's equations for a vacuum, with the displacement current neglected. Only the magnetic field equations are required, and these are

$$\nabla \times \underline{B}^* = 0 \tag{13}$$

$$\nabla \cdot \underline{B}^* = 0 \tag{14}$$

BOUNDARY CONDITIONS

Next we describe the boundary conditions of the interface between the inner and outer fluids. In the real physical situation, there would be a continuous transition of fluid properties across a thin layer separating the two fluids. As is the usual case, we assume the transition is discontinuous, and then allow for a sheet current and a discontinuous jump in the field quantities. The appropriate equations may be integrated across the interface to obtain the following boundary conditions:

$$(P_2^* - P_1^*) + \left[\frac{\left(B_2^*\right)^2}{2\mu_o} - \frac{\left(B_1^*\right)^2}{2\mu_o}\right] = 0 \tag{15}$$

$$\hat{n} \cdot (\underline{V}_2^* - \underline{V}_1^*) = 0 \tag{16}$$

$$\hat{n} \cdot (\underline{B}_2^* - \underline{B}_1^*) = 0 \tag{17}$$

where $\hat{n}$ is a unit vector normal to the interface. Equations (15) to (17) are to be evaluated at the displaced interface. A detailed derivation and discussion of the boundary conditions is given in appendix B.

It is also necessary to have a relation between the velocity field and the displacement of the fluid at the interface. Since all the equations in this report are given in the Eulerian form rather than the Lagrangian form, we cannot, in general, express the various field quantities in terms of a displacement vector. However, when the displacements are infinitesimal, the Lagrangian displacement can be related to the Eulerian velocity. Following reference 9, let the surface of the interface be defined by the function $F(r^*, \theta, z^*, t^*) = 0$. Since a particle lying in the surface must move tangential to the surface, it follows that (ref. 9)

$$\frac{\partial F}{\partial t^*} + (\underline{V}^* \cdot \nabla)F = 0 \tag{18}$$

Let the interface be described by the function

$$F(r^*, \theta, z^*, t^*) = r^* - [\xi^*(\theta, z^*, t^*) + a] \tag{19}$$

where $\xi^*(\theta, z^*, t^*)$ is the radial displacement from the equilibrium position $r^* = a$.

We now obtain a relation between the interface position and the velocity by substituting equation (19) into equation (18). This yields

$$\left(-\frac{\partial \xi^*}{\partial t^*} + V_r^* - \frac{V_\theta^*}{r^*}\frac{\partial \xi^*}{\partial \theta} - V_z^*\frac{\partial \xi^*}{\partial z^*}\right)_{\text{At interface}} = 0 \tag{20}$$

Equation (20) will be used to determine the perturbation velocity in terms of the radial displacement of the interface.

EQUILIBRIUM FLOW

The geometry of the two-fluid wheel flow is shown in figure 1. Cylindrical coordinates are used, with the z-coordinate directed into the paper. The heavy inner fluid is given the subscript 1, and the outer fluid has the subscript 2. The interface between the

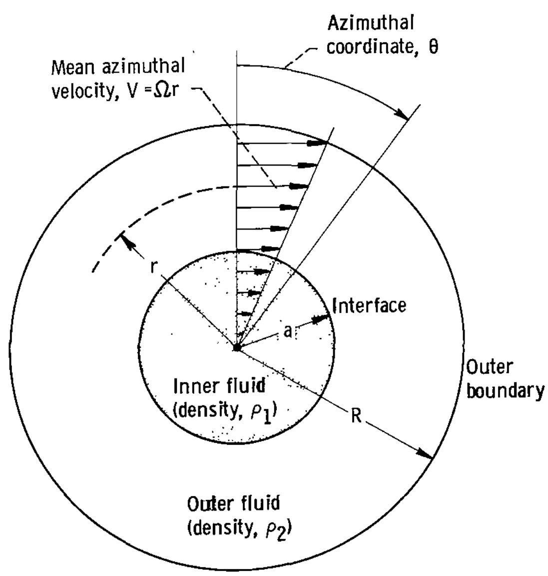


Figure 1. - Geometry of two-fluid wheel flow.

two fluids is at radius a. The outer boundary of the outer fluid is at radius R, which is taken to be infinite in the following analysis. Clearly, the equilibrium flow is unaffected by a cylindrical boundary at a finite radius R, as long as the boundary rotates with the angular velocity of the wheel flow. The reason for letting R be infinite is to simplify the dispersion relation obtained from the perturbation analysis. Since the disturbance is generated at the interface and decays rapidly away from the interface, the outer boundary has little influence on the disturbance growth rates. Reshotko and Monnin (ref. 5) examined the effect of the outermost boundary at $r = R$, and concluded that "For values of ρ_1/ρ_2 and R of interest in wheel flow reactors, the stability characteristics are essentially those of the unbounded configuration."

The inner and outer fluids move in solid-body rotation at the same angular frequency Ω. Thus, the equilibrium flow velocity is azimuthal and depends only on the radial coordinate according to the relation

$$V_\theta^* = \Omega r^* \tag{21}$$

In equilibrium there are no zero-order electric currents; the externally imposed magnetic field is axial and uniform throughout space. Thus, in equilibrium the flow is purely hydrodynamic since the magnetic terms drop out of equation (7). It was shown in reference 5 that V_θ^*, given by equation (21), satisfies the pertinent hydrodynamic momentum and continuity equations (eqs. (7) and (8)). The equilibrium pressure distribution, in the inner and outer fluids, consistent with boundary condition (18), was found to be

$$P_{0,1}^* = P_c^* + \rho_1 \frac{\Omega^2 r^{*2}}{2} \qquad 0 \le r^* \le a \tag{22a}$$

$$P_{0,2}^* = P_c^* + \rho_1 \frac{\Omega^2 a^2}{2} + \rho_2 \Omega^2 \left(\frac{r^{*2} - a^2}{2}\right) \qquad a \le r^* \le R \tag{22b}$$

FORMULATION OF STABILITY PROBLEM

Inner-Fluid Disturbance Equations

The equations governing the infinitesimal disturbances in the inner fluid are obtained from a linearization of equations (7), (8), and (10) about the assumed equilibrium flow. Let $\underline{B}^* = \hat{z}B_0^* + \underline{b}^*$, $\underline{V}^* = \hat{\theta}\underline{r}^*\Omega + \underline{v}^*$, and $P = P_0^* + p^*$. Then the linearized perturbation equations are

Momentum (eq. (7)):

$$\frac{\partial v_r^*}{\partial t^*} + \Omega \frac{\partial v_r^*}{\partial \theta} - 2\Omega v_\theta^* = \frac{B_0^*}{\mu_o \rho} \frac{\partial b_r^*}{\partial z^*} - \frac{1}{\rho} \frac{\partial}{\partial r^*}\left(p^* + \frac{B_0^* b_z^*}{\mu_o}\right) \tag{23}$$

$$\frac{\partial v_\theta^*}{\partial t^*} + \Omega \frac{\partial v_\theta^*}{\partial \theta} + 2\Omega v_r^* = \frac{B_0^*}{\mu_o \rho} \frac{\partial b_\theta^*}{\partial z^*} - \frac{1}{\rho r^*} \frac{\partial}{\partial \theta}\left(p^* + \frac{B_0^* b_z^*}{\mu_o}\right) \tag{24}$$

$$\frac{\partial v_z^*}{\partial t^*} + \Omega \frac{\partial v_z^*}{\partial \theta} = \frac{B_0^*}{\mu_o \rho} \frac{\partial b_z^*}{\partial z^*} - \frac{1}{\rho} \frac{\partial}{\partial z^*}\left(p^* + \frac{B_0^* b_z^*}{\mu_o}\right) \tag{25}$$

Faraday's law for a perfect conductor (eq. (10)):

$$\frac{\partial b_r^*}{\partial t^*} + \Omega \frac{\partial b_r^*}{\partial \theta} = B_0^* \frac{\partial v_r^*}{\partial z^*} \tag{26}$$

$$\frac{\partial b_\theta^*}{\partial t^*} + \Omega \frac{\partial b_\theta^*}{\partial \theta} = B_0^* \frac{\partial v_\theta^*}{\partial z^*} \tag{27}$$

$$\frac{\partial b_z^*}{\partial t^*} + \Omega \frac{\partial b_z^*}{\partial \theta} = B_0^* \frac{\partial v_z^*}{\partial z^*} \tag{28}$$

Continuity (eq. (8)):

$$\frac{\partial v_r^*}{\partial r^*} + \frac{v_r^*}{r^*} + \frac{1}{r^*} \frac{\partial v_\theta^*}{\partial \theta} + \frac{\partial v_z^*}{\partial z^*} = 0 \tag{29}$$

Equations (23) to (29) can be nondimensionalized by using the following definitions:

$$v^* = vv_{ref} \qquad t^* = t\Omega^{-1} \qquad p^* = p\,\frac{B_{ref}^2}{\mu_o}$$

$$B^* = BB_{ref} \qquad \omega^* = \omega\Omega \qquad v_{ref} = a\Omega$$

$$r^* = ar \qquad A_\alpha^2 = \frac{B_{ref}^2}{\mu_o \rho_\alpha v_{ref}^2} \qquad z^* = az$$

where subscript $\alpha = 1$ refers to the inner fluid, and $\alpha = 2$ refers to the outer fluid. The nondimensional form of equations (23) to (29) becomes

$$\frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial \theta} - 2v_\theta = A_1^2 B_0 \frac{\partial b_r}{\partial z} - A_1^2 \frac{\partial}{\partial r}(p + B_0 b_z) \tag{23a}$$

$$\frac{\partial v_\theta}{\partial t} + \frac{\partial v_\theta}{\partial \theta} + 2v_r = A_1^2 B_0 \frac{\partial b_\theta}{\partial z} - \frac{A_1^2}{r} \frac{\partial}{\partial \theta}(p + B_0 b_z) \tag{24a}$$

$$\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial \theta} = A_1^2 B_0 \frac{\partial b_z}{\partial z} - A_1^2 \frac{\partial}{\partial z}(p + B_0 b_z) \tag{25a}$$

$$\frac{\partial b_r}{\partial t} + \frac{\partial b_r}{\partial \theta} = B_0 \frac{\partial v_r}{\partial z} \tag{26a}$$

$$\frac{\partial b_\theta}{\partial t} + \frac{\partial b_\theta}{\partial \theta} = B_0 \frac{\partial v_\theta}{\partial z} \tag{27a}$$

$$\frac{\partial b_z}{\partial t} + \frac{\partial b_z}{\partial \theta} = B_0 \frac{\partial v_z}{\partial z} \tag{28a}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{29a}$$

Since the coefficients of equations (23a) to (29a) are independent of t, θ, and z, the equations allow solutions of the form $q(r)\exp[i(m\theta + kz + \omega t)]$ where $q(r)$ is a complex disturbance amplitude, k and m are the nondimensional axial and azimuthal wave numbers, respectively, and ω is a nondimensional complex frequency. The wave numbers k and m are real, and m is integral. The real part of ω is the nondimensional rotational frequency of the disturbance, while the imaginary part ω_i is the nondimensional growth rate of the disturbance. Disturbances grow, are neutral, or decay according to whether ω_i is negative, zero, or positive, respectively. For the assumed form of the disturbance, equations (23a) to (29a) become

$$icv_r - 2v_\theta = A_1^2 B_0 ikb_r - A_1^2 \left(\frac{dp}{dr} + B_0 \frac{db_z}{dr}\right) \tag{23b}$$

$$icv_\theta + 2v_r = A_1^2 B_0 ikb_\theta - \frac{A_1^2}{r} im\,(p + B_0 b_z) \tag{24b}$$

$$icv_z = -A_1^2 ikp \tag{25b}$$

$$icb_r = B_0 ikv_r \tag{26b}$$

$$icb_\theta = B_0 ikv_\theta \tag{27b}$$

$$icb_z = B_0 ikv_z \tag{28b}$$

$$\frac{dv_r}{dr} + \frac{v_r}{r} + \frac{imv_\theta}{r} + ikv_z = 0 \tag{29b}$$

where $c = \omega + m$, and c is the wave frequency observed in the rotating frame. It is subsequently referred to as the relative frequency. Substitute equations (25b), (26b), and (28b) into equation (23b) and equation (27b) into equation (24b), and let

$$Q_1 = ic\left(1 - \frac{A_1^2 k^2 B_0^2}{c^2}\right) \tag{31}$$

Then

$$Q_1 v_r - 2v_\theta = -\frac{A_1^2 Q_1}{ic} \frac{dp}{dr} \tag{32}$$

$$Q_1 v_\theta + 2v_r = -A_1^2 \frac{m}{rc} Q_1 p \tag{33}$$

Solve equations (32) and (33) for v_r and v_θ in terms of p to obtain

$$v_\theta = \frac{A_1^2}{\delta_1^2 ic}\left(\frac{2}{Q_1} \frac{dp}{dr} - \frac{im}{r} p\right) \tag{34}$$

$$v_r = \frac{-A_1^2}{Q_1^2 \delta_1^2 ic}\left(Q_1^2 \frac{dp}{dr} + \frac{2im}{r} Q_1 p\right) \tag{35}$$

where

$$\delta_1^2 = 1 + \frac{4}{Q_1^2} \tag{36}$$

Substitute equations (25b), (34), and (35) into equation (29b) to obtain

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} - \left(\frac{m^2}{r^2} + k^2\delta_1^2\right)p = 0 \tag{37}$$

The general solution of equation (37) is a linear combination of Bessel functions of the first and second kinds. The form used here is

$$p = \mathscr{A}_1 J_m(ik\delta_1 r) + \mathscr{A}_2 H_m(ik\delta_1 r) \tag{38}$$

The second term blows up at the origin, so $\mathscr{A}_2$ is set equal to zero. The solution in the inner fluid is

$$\boxed{p_1 = \mathscr{A}_1 J_m(ik\delta_1 r)} \tag{39}$$

The boxed equations will be used in the interface boundary condition equations to obtain the dispersion relation. We obtain v_{1r} by substituting p_1 for p into equation (35).

$$\boxed{v_{1r} = \frac{-\mathscr{A}_1 A_1^2}{Q_1\delta_1^2 ic}\left\{Q_1 \frac{d}{dr}\left[J_m(ik\delta_1 r)\right] + \frac{2im}{r} J_m(ik\delta_1 r)\right\}} \tag{40}$$

The magnetic field component b_{1r} is obtained by substituting equation (40) into equation (26b).

$$\boxed{b_{1r} = -\mathscr{A}_1 \frac{B_0 k A_1^2}{Q_1\delta_1^2 ic^2}\left\{Q_1 \frac{d}{dr}\left[J_m(ik\delta_1 r)\right] + \frac{2im}{r} J_m(ik\delta_1 r)\right\}} \tag{41}$$

Finally, b_{1z} is obtained by combining equations (25b), (28b), and (39).

$$\boxed{b_{1z} = -\mathscr{A}_1 \frac{B_0 k^2}{c^2} A_1^2 J_m(ik\delta r)} \tag{42}$$

Outer-Fluid Disturbance Equations

Linearization of equations (11) to (14) about the equilibrium flow yields the disturbance equations for the outer fluid.

Momentum (eq. (11)):

$$\frac{\partial v_r^*}{\partial t^*} + \Omega \frac{\partial v_r^*}{\partial \theta} - 2\Omega v_\theta^* = -\frac{1}{\rho} \frac{\partial p^*}{\partial r^*} \tag{43}$$

$$\frac{\partial v_\theta^*}{\partial t^*} + \Omega \frac{\partial v_\theta^*}{\partial \theta} + 2\Omega v_r^* = -\frac{1}{\rho r^*} \frac{\partial p^*}{\partial \theta} \tag{44}$$

$$\frac{\partial v_z^*}{\partial t^*} + \Omega \frac{\partial v_z^*}{\partial \theta} = -\frac{1}{\rho} \frac{\partial p^*}{\partial z^*} \tag{45}$$

Continuity (eq. (12)):

$$\frac{\partial v_r^*}{\partial r^*} + \frac{v_r^*}{r^*} + \frac{1}{r^*} \frac{\partial v_\theta^*}{\partial \theta} + \frac{\partial v_z^*}{\partial z^*} = 0 \tag{46}$$

Div $\underline{B} = 0$ (eq. (13)):

$$\frac{\partial b_r^*}{\partial r^*} + \frac{b_r^*}{r^*} + \frac{1}{r^*} \frac{\partial b_\theta^*}{\partial \theta} + \frac{\partial b_z^*}{\partial z^*} = 0 \tag{47}$$

Curl $\underline{B} = 0$ (eq. (14)):

$$\frac{1}{r^*} \frac{\partial b_z^*}{\partial \theta} - \frac{\partial b_\theta^*}{\partial z^*} = 0 \tag{48}$$

$$\frac{\partial b_z^*}{\partial r^*} - \frac{\partial b_r^*}{\partial z^*} = 0 \tag{49}$$

$$\frac{\partial b_\theta^*}{\partial r^*} + \frac{b_\theta^*}{r^*} - \frac{1}{r^*} \frac{\partial b_r^*}{\partial \theta} = 0 \tag{50}$$

The nondimensional forms of equations (43) to (50) are

$$\frac{\partial v_r}{\partial t} + \frac{\partial v_r}{\partial \theta} - 2v_\theta = -A_2^2 \frac{\partial p}{\partial r} \tag{51a}$$

$$\frac{\partial v_\theta}{\partial t} + \frac{\partial v_\theta}{\partial \theta} + 2v_r = - \frac{A_2^2}{r} \frac{\partial p}{\partial \theta} \tag{52a}$$

$$\frac{\partial v_z}{\partial t} + \frac{\partial v_z}{\partial \theta} = -A_2^2 \frac{\partial p}{\partial z} \tag{53a}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{54a}$$

$$\frac{\partial b_r}{\partial r} + \frac{b_r}{r} + \frac{1}{r} \frac{\partial b_\theta}{\partial \theta} + \frac{\partial b_z}{\partial z} = 0 \tag{55a}$$

$$\frac{1}{r} \frac{\partial b_z}{\partial \theta} - \frac{\partial b_\theta}{\partial z} = 0 \tag{56a}$$

$$\frac{\partial b_z}{\partial r} - \frac{\partial b_r}{\partial z} = 0 \tag{57a}$$

$$\frac{\partial b_\theta}{\partial r} + \frac{b_\theta}{r} - \frac{1}{r} \frac{\partial b_r}{\partial \theta} = 0 \tag{58a}$$

As before, the equations allow solutions of the form $q(r)\exp[i(m\theta + kz + \omega t)]$. Equations (51a), (52a), and (53a) become, respectively,

$$Q_2 v_r - 2v_\theta = -A_2^2 \frac{dp}{dr} \tag{51b}$$

$$Q_2 v_\theta + 2v_r = -A_2^2 \frac{imp}{r} \tag{52b}$$

$$Q_2 v_z = -A_2^2 ikp \tag{53b}$$

where $Q_2 = ic$. And equation (54a) becomes

$$\frac{dv_r}{dr} + \frac{v_r}{r} + \frac{im}{r} v_\theta + ikv_z = 0 \tag{54b}$$

Solve equations (51b) and (52b) for v_r and v_θ in terms of p to obtain

$$v_\theta = \frac{A_2^2}{ic\delta_2^2 Q_2}\left(2\frac{dp}{dr} - \frac{Q_2 imp}{r}\right) \tag{59}$$

$$v_r = -\frac{A_2^2}{ic\delta_2^2 Q_2}\left(Q_2\frac{dp}{dr} + \frac{2imp}{r}\right) \tag{60}$$

where $\delta_2^2 = 1 + 4/Q_2^2$.

Substitute v_θ from equation (59), v_r from equation (60), and v_z from equation (53b) into equation (54b) to obtain the differential equation for the perturbation pressure in the outer fluid

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} - \left(\frac{m^2}{r^2} + k^2\delta_2^2\right)p = 0 \tag{62}$$

This differential equation is of the same form as equation (37). The solutions to equation (62) are therefore

$$p = \mathscr{A}_1 J_m(ik\delta_2 r) + \mathscr{A}_2 H_m(ik\delta_2 r) \tag{63}$$

where H_m is the Hankel function of the first kind. It is usually denoted by $H_m^{(1)}$, but the superscript was removed to simplify notation in this report. As $r \to \infty$, the first term in equation (63) blows up, so $\mathscr{A}_1$ is set equal to zero. The solution for the perturbation pressure term in the outer fluid is therefore

$$\boxed{p_2 = \mathscr{A}_2 H_m(ik\delta_2 r)} \tag{64}$$

The perturbation velocity term v_r is obtained by substituting p_2 for p in equation (60), then

$$\boxed{v_{2r} = -\frac{\mathscr{A}_2 A_2^2}{ic\delta_2^2 Q_2}\left\{Q_2 \frac{d}{dr}\left[H_m(ik\delta_2 r)\right] + \frac{2im}{r} H_m(ik\delta_2 r)\right\}} \tag{65}$$

We obtain b_r and b_z from equations (55a), (56a), and (57a) which become, respectively, after assuming all first-order quantities vary as $q(r)\exp[i(m\theta + kz + \omega t)]$

$$\frac{db_r}{dr} + \frac{b_r}{r} + \frac{im}{r} b_\theta + ikb_z = 0 \tag{55b}$$

$$\frac{im}{r} b_z - ikb_\theta = 0 \tag{56b}$$

$$\frac{db_z}{dr} - ikb_r = 0 \tag{57b}$$

Substituting b_θ from equation (56b) and b_r from equation (57b) into equation (55b) yields the differential equation for b_z in the outer region

$$\frac{d^2 b_z}{dr^2} + \frac{1}{r}\frac{db_z}{dr} - \left(\frac{m^2}{r^2} + k^2\right) b_z = 0 \tag{66}$$

Again, the general solution is a linear combination of Bessel functions of the first and second kinds

$$b_z = \mathscr{B}_1 J_m(ikr) + \mathscr{B}_2 H_m(ikr) \tag{67}$$

As $r \to \infty$ the perturbation field must remain finite, so $\mathscr{B}_1$ is set equal to zero. Then

$$\boxed{b_{2z} = \mathscr{B}_2 H_m(ikr)} \tag{68}$$

Substituting b_z from equation (68) into equation (57b) yields

$$\boxed{b_{2r} = \mathscr{B}_2 \frac{1}{ik} \frac{d}{dr}\left[H_m(ikr)\right]} \tag{69}$$

Application of Boundary Conditions at Interface

The boundary conditions requiring all perturbation fields to be finite at $r = 0$ and $r = \infty$ have already been applied. The remaining conditions, applicable at the interface, are given by equations (15) to (17). The surface normal can be written as

$$\hat{n} = \hat{r} + \underline{\boldsymbol{n}}$$

The zero-order surface normal is the unit vector in the radial direction $\hat{r}$. The quantity $\underline{\boldsymbol{n}}$ is a first-order perturbation of the unit vector. The first-order boundary conditions on the normal components of velocity and magnetic field at the interface are obtained from equations (16) and (17) as

$$\hat{r} \cdot (\underline{v}_2 - \underline{v}_1) = v_{2r} - v_{1r} = 0 \qquad \text{at } r = 1 \tag{70}$$

$$\hat{r} \cdot (\underline{b}_2 - \underline{b}_1) = b_{2r} - b_{1r} = 0 \qquad \text{at } r = 1 \tag{71}$$

Strictly speaking, these two boundary conditions should be evaluated at the displaced interface. But, to within first order, equations (16) and (17) may be evaluated at the equilibrium interface position, $r^* = a$ or equivalently $r = 1$.

Substituting equations (35), (39), and (65) into equation (70) gives the condition of continuity of radial velocity at $r = 1$,

$$\frac{A_1^2}{Q_1^2\delta_1^2}\left[Q_1^2 J'_m(ik\delta_1) + 2imQ_1 J_m(ik\delta_1)\right]\mathscr{A}_1 = \frac{A_2^2}{Q_2^2\delta_2^2}\left[Q_2^2 H'_m(ik\delta_2) + 2imQ_2 H_m(ik\delta_2)\right]\mathscr{A}_2 \tag{72}$$

where prime denotes differentiation with respect to r.

When equations (26b), (35), (39), and (69) are substituted in equation (71), the condition for continuity of radial magnetic field at $r = 1$ becomes

$$\frac{ikB_0}{c^2 Q_1^2\delta_1^2}\mathscr{A}_1\left[Q_1^2 J'_m(ik\delta_1) + 2imQ_1 J_m(ik\delta_1)\right] = \frac{\mathscr{B}_2}{ik} H'_m(ik) \tag{73}$$

The boundary condition in equation (15) states that the normal force is continuous across the displaced interface. Writing out equation (15) to all orders results in the following equation, in physical variables:

$$\left(P^*_{0,2} + p^*_2\right) - \left(P^*_{0,1} + p^*_1\right) + \left(\frac{\underline{B}^*_2 \cdot \underline{B}^*_2}{2\mu_o}\right) - \left(\frac{\underline{B}^*_1 \cdot \underline{B}^*_1}{2\mu_o}\right) = 0 \tag{74}$$

where all quantities in equation (74) are evaluated at $r^* = a + \xi^*$. Now

$$\underline{B}^*_\alpha \cdot \underline{B}^*_\alpha = \left(\underline{B}^*_0 + \underline{b}^*_{\alpha z}\right) \cdot \left(\underline{B}^*_0 + \underline{b}^*_{\alpha z}\right) = B_0^{*2} + 2B^*_0 b^*_{\alpha z} + b^{*2}_{\alpha z}$$

We may now write out all terms in equation (74), which yields

$$\left[P^*_0 + \rho_1\frac{\Omega^2 a^2}{2} + \rho_2\Omega^2\frac{(a+\xi^*)^2 - a^2}{2} + p^*_2(\xi^*)\right] - \left[P^*_0 + \rho_1\Omega^2\frac{(a+\xi^*)^2}{2} + p^*_1(\xi^*)\right]$$

$$+\left[\frac{B_0^{*2}}{2\mu_o} + \frac{B^*_0}{\mu_o}b^*_{2z}(\xi^*) + \frac{b^{*2}_{2z}(\xi^*)}{2\mu_o}\right] - \left[\frac{B_0^{*2}}{2\mu_o} + \frac{B^*_0}{\mu_o}b^*_{1z}(\xi^*) + \frac{b^{*2}_{1z}(\xi^*)}{2\mu_o}\right] = 0 \tag{75}$$

The radial displacement ξ^* is obtained from equation (20) where

$$V^*_r = v^*_r$$

$$V_\theta^* = r^*\Omega + v_\theta^*$$

$$V_z^* = v_z^*$$

Substituting these into equation (20) and treating ξ^* as first order yields to first order

$$\left(-\frac{\partial\xi^*}{\partial t^*} + v_r^* - \Omega\frac{\partial\xi^*}{\partial\theta}\right)_{r^*=a} = 0$$

Assuming that ξ^* goes as $\exp[i(m\theta + \omega^*t^* + k^*z^*)]$, we get from the last equation

$$-i\omega^*\xi^*(a) + v_r^*(a) - im\Omega\xi^* = 0$$

And, after rearranging,

$$\xi^*(a) = \frac{v_r^*(a)}{i(\omega^* + m\Omega)}$$

or

$$\frac{\xi^*(a)}{a} = \frac{v_r^*(a)}{i\Omega a\left(\frac{\omega^*}{\Omega} + m\right)} = -i\frac{v_r(1)}{c}$$

Thus, in dimensionless form the displacement is

$$\xi(1) = -\frac{iv_r(1)}{c} \tag{76}$$

Retaining only first-order terms in equation (75) leaves

$$\rho_2\Omega^2 a\xi^* + p_2^*(\xi^*) - \rho_1\Omega^2 a\xi^* - p_1^*(\xi^*) + \frac{B_0^*}{\mu_o} b_{2z}^*(\xi^*) - \frac{B_0^*}{\mu_o} b_{1z}^*(\xi^*) = 0 \tag{77}$$

Nondimensionalizing equation (77) gives

$$\xi(1) + A_1^2 p_1(1) + A_1^2 B_0 b_{1z}(1) = \frac{\rho_2}{\rho_1}\xi(1) + \frac{\rho_2}{\rho_1}A_2^2 p_2(1) + \frac{\rho_2}{\rho_1}A_2^2 B_0 b_{2z}(1) \tag{78}$$

It follows from the definition of A_α^2 that $A_1^2 = (\rho_2/\rho_1)\,A_2^2$. If we choose V_{ref} so that $A_1^2 = 1$, equation (78) becomes

$$\xi(1) + p_1(1) + B_0 b_{1z}(1) = \frac{\rho_2}{\rho_1}\xi(1) + p_2(1) + B_0 b_{2z}(1) \tag{79}$$

Substituting into equation (79) the quantities $\xi(1)$ from equation (76), $p_1(1)$ from equation (39), $b_{1z}(1)$ from equation (42), $p_2(1)$ from equation (64), and $b_{2z}(1)$ from equation (68) results in the continuity of normal force boundary condition

$$\frac{\mathscr{A}_1}{c^2 Q_1^2 \delta_1^2}\left[Q_1^2 J'_m(ik\delta_1) + 2imQ_1 J_m(ik\delta_1)\right] + \mathscr{A}_1 J_m(ik\delta_1)$$

$$- \frac{k^2 B_0^2}{c^2}\mathscr{A}_1 J_m(ik\delta_1) = \mathscr{B}_2 B_0 H'_m(ik) + \mathscr{A}_2 H_m(ik\delta_2)$$

$$+ \frac{\mathscr{A}_2}{c^2 Q_2^2 \delta_2^2}\left[Q_2^2 H_m(ik\delta_2) + 2imQ_2 H_m(ik\delta_2)\right] \tag{80}$$

DISPERSION RELATION

From equation (72) for continuity of radial velocity,

$$\mathscr{A}_1 = \left(\frac{\dfrac{\rho_1}{\rho_2 Q_2^2 \delta_2^2}\mathscr{H}_m}{\dfrac{1}{Q_1^2 \delta_1^2}\mathscr{J}_m}\right)\mathscr{A}_2 \tag{81}$$

where

$$\mathscr{H}_m = Q_2^2 H_m'(ik\delta_2) + 2imQ_2 H_m(ik\delta_2) \tag{82}$$

$$\mathscr{J}_m = Q_1^2 J_m'(ik\delta_1) + 2imQ_1 J_m(ik\delta_1) \tag{83}$$

From equation (73) for continuity of radial magnetic field,

$$\mathscr{B}_2 = -\frac{\mathscr{A}_1 k^2 B_0}{c^2 Q_1^2 \delta_1^2} \frac{\mathscr{J}_m}{H_m'(ik)} \tag{84}$$

Substitute equations (81) and (84) into equation (80) to obtain

$$\frac{\left(\dfrac{\rho_1}{\rho_2 Q_2^2 \delta_2^2}\right)\mathscr{H}_m}{\left(\dfrac{1}{Q_1^2 \delta_1^2}\right)\mathscr{J}_m}\left[\frac{\mathscr{J}_m}{c^2 Q_1^2 \delta_1^2} + J_m(ik\delta_1) - \frac{k^2 B_0^2}{c^2} J_m(ik\delta_1) + \frac{k^2 B_0^2 \mathscr{J}_m H_m(ik)}{c^2 Q_1^2 \delta_1^2 H_m'(ik)}\right]$$

$$= H_m(ik\delta_2) + \frac{1}{c^2 Q_2^2 \delta_2^2} \mathscr{H}_m \tag{85}$$

Using the definitions of Q_α, $\mathscr{J}_m$, and $\mathscr{H}_m$ and rearranging terms give

$$\left[\frac{\delta_1^2}{D} - \frac{\left(\dfrac{\rho_2}{\rho_1}\right)\delta_2^2}{F}\right] c^2 = \frac{\rho_2}{\rho_1} - 1 + k^2 B_0^2 \left(\frac{\delta_1^2}{D} - \frac{1}{G}\right) \tag{86}$$

where, after setting $A_1 = 1$,

$$D \equiv \frac{J_m'(ik\delta_1)}{J_m(ik\delta_1)} + \frac{2im}{Q_1} \tag{87}$$

$$G \equiv \frac{H_m'(ik)}{H_m(ik)} \tag{88}$$

$$Q_2 \equiv ic \tag{89}$$

$$\delta_2^2 \equiv 1 - \frac{4}{c^2} \tag{90}$$

$$F \equiv \frac{H_m'(ik\delta_2)}{H_m(ik\delta_2)} + \frac{2im}{Q_2} \tag{91}$$

$$Q_1 \equiv ic\left(1 - \frac{k^2B_0^2}{c^2}\right) \tag{92}$$

$$\delta_1^2 \equiv 1 - \frac{4}{c^2\left(1 - \frac{k^2B_0^2}{c^2}\right)^2} \tag{93}$$

$$c \equiv \omega + m \tag{94}$$

Prime denotes differentiation with respect to r. Thus,

$$J_m'(ik\delta_1) = \left[\frac{d}{dr} J_m(ik\delta_1 r)\right]_{r=1}$$

and

$$H_m'(ik) = \left[\frac{d}{dr} H_m(ikr)\right]_{r=1}$$

Equation (86), with the definitions following it, is the dispersion relation. It is a complex equation since ω, and therefore c, is complex. In fact, the Bessel functions have complex arguments because of the δ_α factors. The dispersion relation was solved numerically by using an iterative technique on an IBM 7094 computer.

The dispersion relation has multiple roots of which some are nongrowing Alfven waves. A technique was incorporated into the program to eliminate these uninteresting solutions.

APPROXIMATE SOLUTIONS TO DISPERSION RELATION

Long Wavelength Limit

The long wavelength solutions $k \to 0$ to the dispersion relation have been determined in appendix C. The results are

$$\mathrm{Re}(c) = \frac{1}{\mathscr{R}} \tag{95}$$

or

$$\mathrm{Re}(\omega) = -m + \frac{1}{\mathscr{R}} \qquad m > 0 \tag{96}$$

and

$$\mathrm{Im}(c) = -\sqrt{\mathscr{R}m - 1} \qquad m > 0 \tag{97}$$

where

$$\mathscr{R} = \frac{\dfrac{\rho_1}{\rho_2} + 1}{\dfrac{\rho_1}{\rho_2} - 1} \tag{98}$$

These solutions were used as initial points in the iterative computer solutions.

Solutions Where Wave Frequency c = 0

The values of k and B_0 for which c goes to zero are determined in appendix D. Having selected a value of B_0 or k, the following relation gives the value of k or B_0 for which c = 0:

$$\frac{k^2 B_0^2}{1 - \frac{\rho_2}{\rho_1}} = k^2 I_{m-1}(k)K_{m-1}(k) + m\left[I_{m-1}(k)K_m(k) - I_m(k)K_{m-1}(k)\right] - m^2 I_m(k)K_m(k) \quad (99)$$

Equation (99) is further reduced to the limiting cases of $k \to 0$ and $k \to \infty$ in appendix D. Equation (99) was used as a check of the computer solutions.

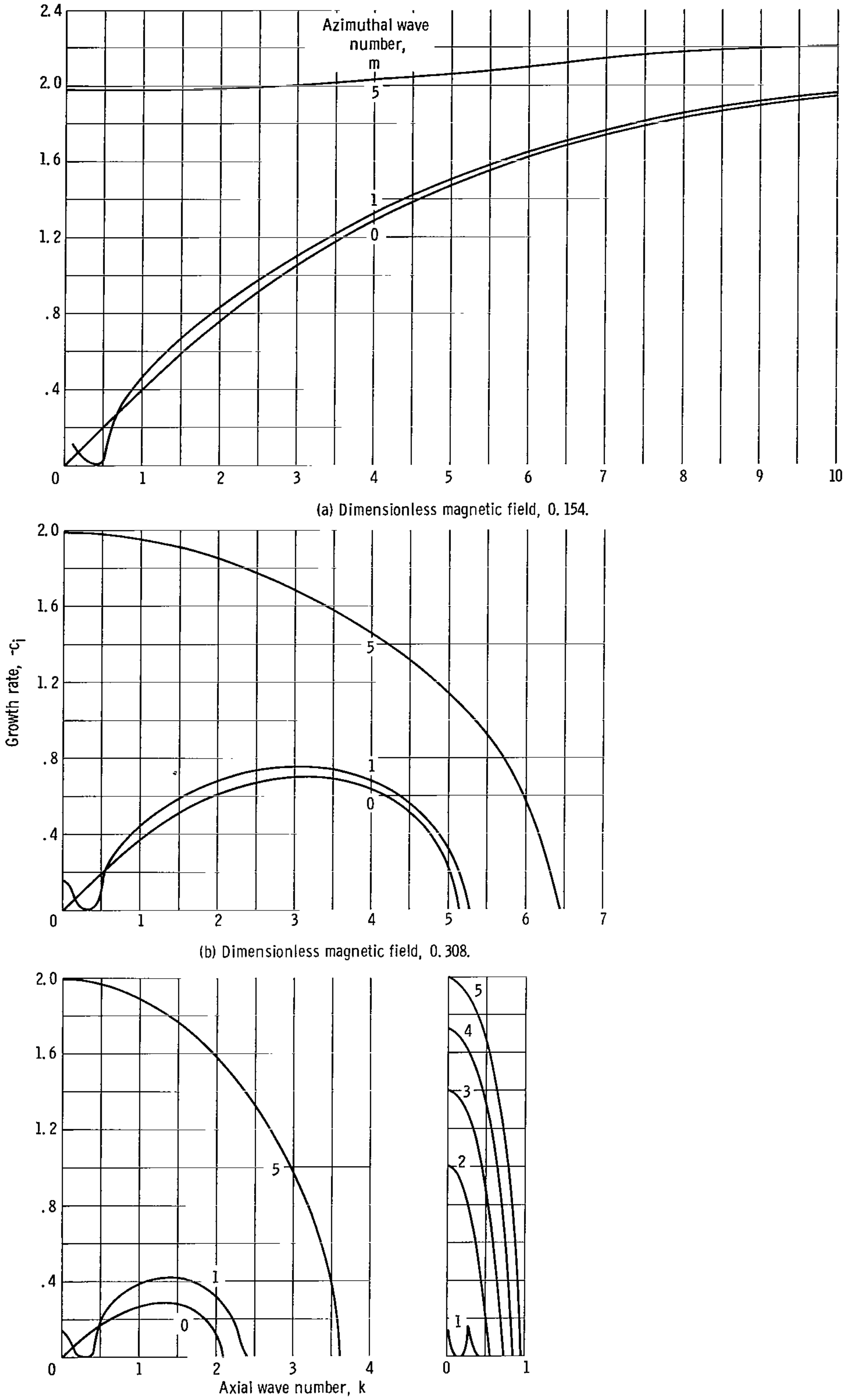
DISCUSSION

Although many stable modes of oscillation were found for the dispersion relation, only those solutions which indicate disturbances that grow with time are discussed. After the discussion of solutions to the dispersion equation, we explore some of the limitations of the normal mode approach to the stability problem and suggest some ways to interpret the results. Next, some physical implications about the magnetically stabilized plasma are considered. Potential methods are suggested to avoid the instabilities predicted by the present analysis.

Presented in figures 2 and 3 are curves of the imaginary and real parts, respectively, of the relative wave frequency c as a function of the axial wave number k. In these plots, the azimuthal wave number is a parameter, and results are presented for four different values of the externally imposed magnetic field. These curves are solutions to equation (86), the dispersion relation.

Typical fluid properties for the wheel-flow reactor (ref. 10) were used to establish the range of dimensionless magnetic fields found in figures 2 and 3. An angular velocity of 200 radians per second was selected to provide a plasma Mach number of less than 0.1 at the interface. A density ratio ρ_1/ρ_2 of about 100, the largest envisioned, was chosen for the calculations because it is the most unstable configuration. The temperature chosen was 40 000 K, and the pressure was of the order of 100 atmospheres. The reactor radius was 1 meter, and the density ρ_1 was 10.3 kilograms per cubic meter.

As shown in figure 2 in all but the weakest magnetic field case, the growth rate c_i decreases to zero at a finite value of the axial wave number for a given value of m. For a given m and a given magnetic field, both the real (fig. 3) and imaginary (fig. 2)



(a) Dimensionless magnetic field, 0.154.

(b) Dimensionless magnetic field, 0.308.

(c) Dimensionless magnetic field, 0.462.

(d) Dimensionless magnetic field, 1.54.

Figure 2. - Disturbance growth rate as function of axial wave number with azimuthal wave number as parameter.

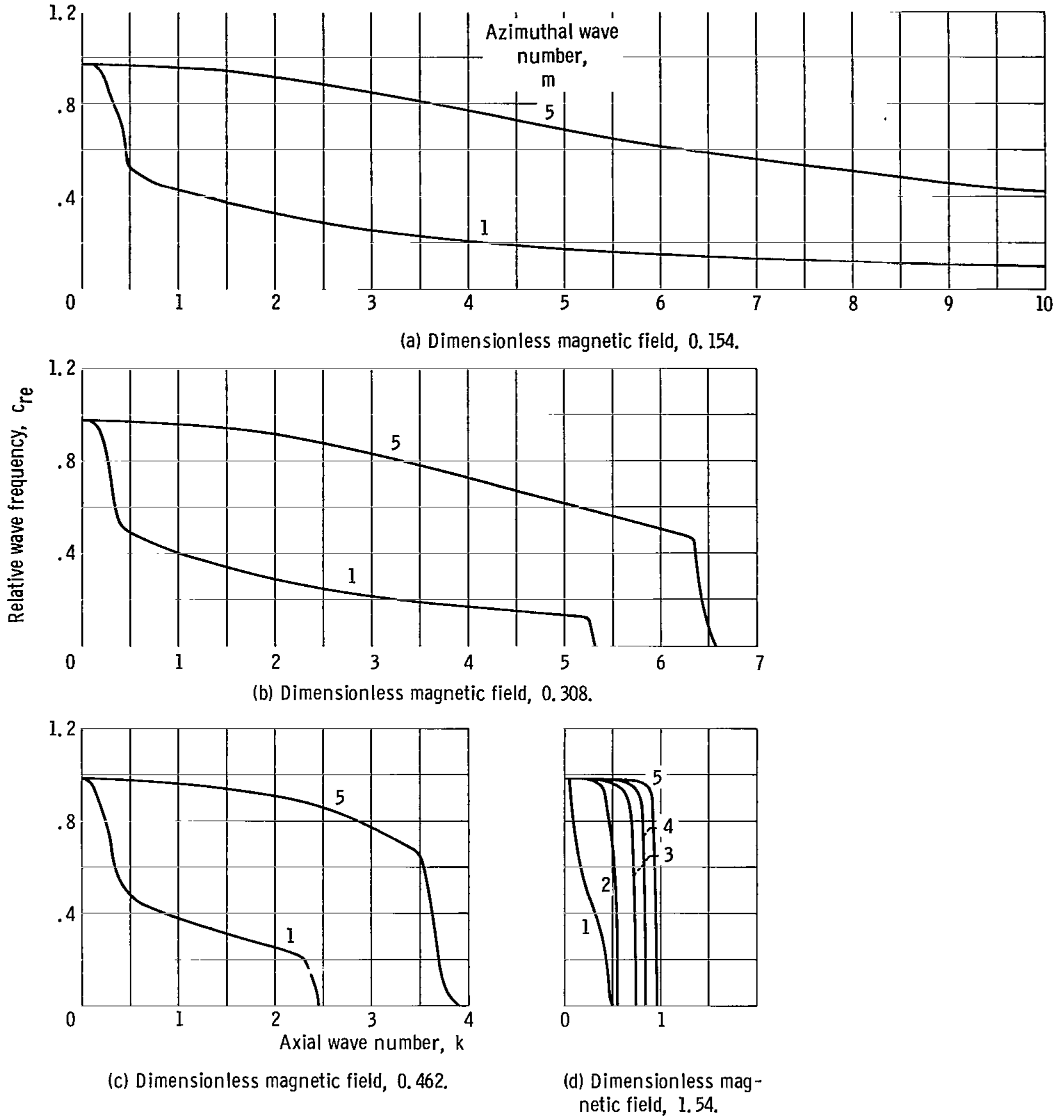


Figure 3. - Disturbance relative wave frequency as function of axial wave number with azimuthal wave number as parameter.

parts of c must go to zero at the same value of k. Because of the scales used in the figures, it appears on some plots that c_i and c_{re} go to zero at different values of k, but the computer results verify that both the real and imaginary parts go to zero at the same value of k. The value of k at which c goes to zero is referred to as the critical wave number $k_{critical}$. Beyond $k_{critical}$, there are no solutions that correspond to a continuation of the curves presented in figures 2 and 3.

In the hydrodynamic case (ref. 5), the flow was found to be always unstable. However, figure 2 reveals that the effect of the magnetic field is to make the system stable to short wavelength disturbances, $\lambda < 2\pi/k_{critical}$. The short wavelengths are stabilized because the magnetic field lines, which are "frozen" into the fluid, undergo a distortion or "stretching" when the interface is perturbed. It requires energy to stretch the field lines. Thus, the energy that would normally be available to drive the instability is absorbed in the stretching process. In the hydrodynamic Rayleigh-Taylor problem, surface tension provides the "stretching" mechanism that stabilizes the short-wavelength disturbances.

Figure 2 also reveals that for a given value of k, the growth rate increases as the azimuthal wave number increases. This suggests that smaller chunks of the heavy fluid have the most likelihood of tearing off at the interface. In figure 2, the $m = 1$ curves have a dip in them near $k = 0$. Although it is not obvious from the figure, the growth rate is finite at the minimum of the dip. This same peculiar dip occurred in reference 5, and we have no good physical explanation for its occurrence.

We note from figure 2 that the $k = 0$ disturbances (infinitely long wavelength) are unstable for all azimuthal wave numbers except $m = 0$. (When $m = 0$ and $k = 0$, there is no disturbance on the flow.) These unstable modes are the pure flute modes. They do not distort the magnetic field and, hence, are not stabilized by it. When $k = 0$, the dispersion relation reduces to the simple form given in equations (95) to (98). From equation (95) and from figure 3, note that c_{re} reduces to $1/\mathscr{R}$, when $k = 0$. Thus, to an observer moving with the rotating frame, all flute modes oscillate with the same frequency.

In figure 4, the value of magnetic field for which c goes to zero is presented as a function of $k_{critical}$, with m as a parameter. These are plots of equation (99) in which the dispersion equation was solved subject to the requirement $c = 0$. These curves have a simple interpretation: for those combinations of magnetic field and axial wave number that lie above the curve, the plasma is stable, while it is unstable if the combination falls below the curve. Information found from figures 2 to 4 and equation (C9) indicates that no matter what value of magnetic field is chosen, a sufficiently high value of m will make the system unstable to long-wavelength disturbances. But when the value of m becomes so large that the azimuthal wavelength is comparable with the boundary layer thickness, the analysis breaks down. The analysis implies that even the idealized concept of a two-fluid wheel-flow device is impractical unless some way is found to place restrictions on the axial wavelength and azimuthal wave numbers.

The dispersion relation was obtained from the linearized perturbation analysis about an assumed equilibrium flow. This suggests several interesting limitations to the analysis. We shall address ourselves to three of these problem areas.

First, it was assumed in the theoretical analysis that a particular equilibrium flow could be established. Experimentally this may not be possible. However, the stability analysis gives us some insight about the assumed equilibrium flow. If in establishing

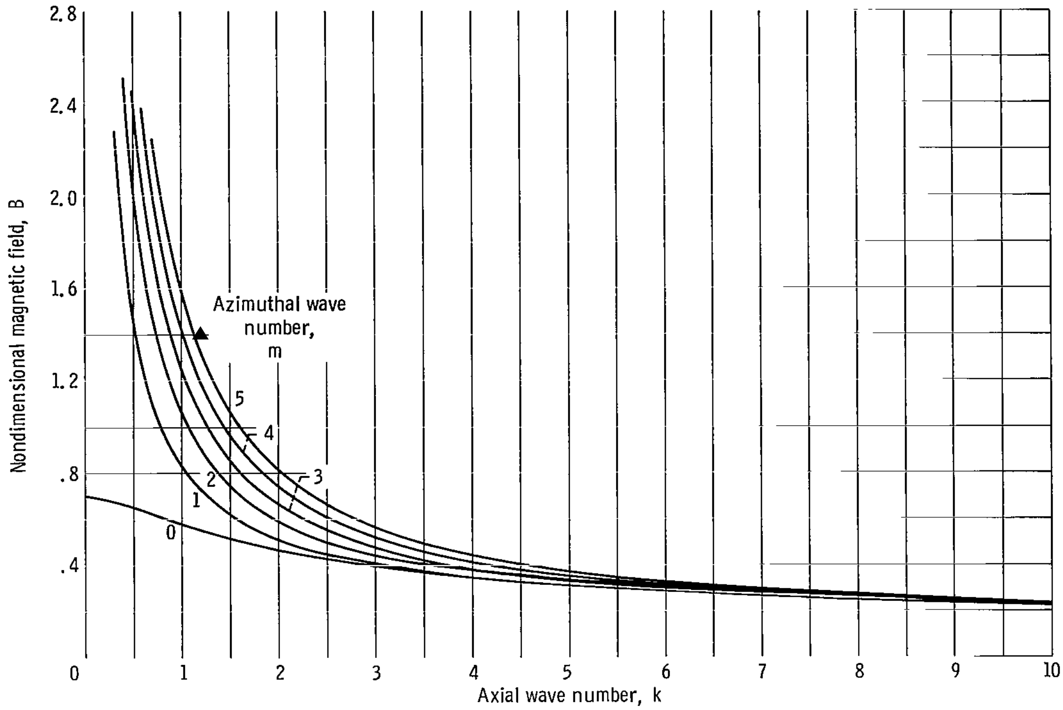


Figure 4. - Neutral stability curves for various azimuthal wave numbers. (Flow is stable above curve and unstable below.)

the equilibrium state the flow must pass through some of the perturbations that are predicted to be unstable, we can be certain of great difficulties, if not the impossibility, in reaching the equilibrium situation. However, the analysis suggests that if the disturbances could be restricted to short axial wavelengths, the equilibrium flow might be established. This might be accomplished by keeping the axial length of the machine short.

The second limitation is concerned with the assumption of small amplitude disturbances and the attendant linearized analysis. Unstable solutions of the linearized problem represent only the first stage of growth, or of departure, of the system from the equilibrium state. As the disturbance amplitude becomes large, the flow should be described by nonlinear differential equations. It is possible for nonlinear effects to limit the disturbance amplitudes and produce nongrowing, finite amplitude disturbances. We do not expect this in the present case. The state of minimum potential energy for the wheel-flow problem corresponds to the heavier fluid being at the outside with the lighter fluid at the center. The nonlinear effects may control the growth rate of the instabilities, but should not terminate the growth. Even in the case of the short-wavelength disturbances, the predicted stability applies only to small amplitudes. For large amplitude disturbances at these wavelengths, the nonlinear effects could produce a positive growth rate.

The linearized perturbation analysis is useful in two respects. First, of course, it warns of regions where the system is unstable even to the infinitesimal perturbations. Second, it suggests which system parameters might be manipulated to maintain or improve the stability in regions where the infinitesimal perturbations do not grow. The linear analysis shows that improved stabilization is obtained by increasing the magnetic field and restricting the disturbances to short wavelengths. Stabilization against large amplitude disturbances might be expected to improve with use of even larger magnetic fields and shorter wavelengths. The utility of this interpretation was demonstrated in reference 11, where the stability of a liquid-mercury - air interface was experimentally investigated. In these tests, mercury in glass tubes was subjected to vibrations of various frequencies and amplitudes, and the limits of interfacial stability were noted. As the amplitude and frequency of vibration were increased, a smaller tube diameter was needed to stabilize the interface. This was consistent with the Rayleigh-Taylor theory, with surface tension effects included, which predicts that the interface is stable against short-wavelength disturbances.

Finally, we want to know how the results of this report would be altered by the inclusion of finite electrical conductivity. Kruskal and Schwarzschild (ref. 8) obtained an approximate solution to a simple hydromagnetic problem with finite conductivity and small perturbations. They compared the results with the infinite conductivity case. They concluded that the essential features of the large but finite conductivity problem were fairly represented if one assumed infinite conductivity, while allowing for electric sheet currents and electric sheet charges at the plasma surface.

Intuitively, it seems that the result of finite, but large, conductivity would be somewhat larger growth rates and less stability at the shorter wavelengths than predicted by the infinite conductivity case. The effect of finite conductivity is to allow the fluid to slip through the magnetic field, without a proportional distortion of the field. The energy stored in the distorted field is less than it would be for infinite conductivity. Finite conductivity is accompanied by a dissipative effect because a fraction of the field energy created by the magnetic field distortions is converted into joule heat. By contrast, the energy-storage capacity of surface tension is independent of the growth rate because it is not accompanied by any dissipation of energy.

The importance of the finite conductivity can be deduced from dimensional analysis. When the fluid has finite conductivity, equation (10) is modified by the inclusion of another term, becoming

$$\frac{1}{\sigma\mu_0} \nabla^2 \underline{B}^* + \nabla \times (\underline{V}^* \times \underline{B}^*) = \frac{\partial \underline{B}^*}{\partial t^*} \tag{100}$$

When this equation is nondimensionalized, we get

$$\frac{1}{\Omega a^2 \sigma \mu_o} \nabla^2 \underline{B} + \nabla \times (\underline{V} \times \underline{B}) = \frac{\partial \underline{B}}{\partial t} \qquad (101)$$

This equation can be reduced to equation (10) if $\Omega a^2 \sigma \mu_o >> 1$. The dimensionless quantity $\Omega a^2 \sigma \mu_o$ is the magnetic Reynolds number. When the magnetic Reynolds number is large, the field restrains the fluid as it does in the infinite conductivity case, where fluid stays "frozen" to the field lines. The criterion of large magnetic Reynolds number can be rewritten as a requirement on the electrical conductivity, namely

$$\sigma >> \frac{1}{\Omega a^2 \mu_o}$$

Choosing $\Omega = 200$ radians per second, $a = 1$ meter, and $\mu_o = 4\pi \times 10^{-7}$ henry per meter, we find that $\sigma >> 4000$ mho per meter is the requirement for the infinite conductivity assumption to apply.

For the proposed wheel-flow reactor, the plasma has a temperature of about 40 000 K and a pressure of a few hundred atmospheres. To estimate the electrical conductivity, we assume that the plasma is fully ionized and use the formula given by Spitzer (ref. 12, eq. (5-37)) for electrical conductivity. In deriving the expression for conductivity, Spitzer neglected terms of the order $1/\ln \Lambda$, where $\ln \Lambda$ is the Coulomb logarithm. In the proposed wheel-flow reactor, $\ln \Lambda$ is of the order of 2.5 to 5. Thus, the conductivity can be estimated to an accuracy of 20 to 40 percent at best. The presence of multiply ionized ions would increase the calculated conductivity by slightly less than a factor of 2, while partial ionization would decrease the conductivity due to electron-neutral collisions and recombination. But the calculated conductivity should be valid to within an order of magnitude, which is sufficient for our purposes.

Based on Spitzer's formula, the conductivity is between 2500 and 5000 mho per meter. Hence, finite conductivity effects will play an important role in the proposed wheel-flow reactor. The flow could, therefore, be substantially more unstable than predicted by the infinite conductivity analysis.

In contrast to the wheel-flow reactor plasma, liquid metals have much higher conductivities. Typical conductivities for the liquid metals sodium, potassium, and mercury are respectively 10^7, 7×10^6, and 10^6 mho per meter. These conductivities are well above 4000 mho per meter, so the infinite conductivity approximation should apply to two-fluid wheel-flow configurations, where the inner fluid is one of these liquid metals.

A second criterion for the infinite conductivity analysis to apply involves the characteristic time in which magnetic field perturbations are dissipated by joule heating.

When the conductivity is so low that the diffusion term in equation (100) dominates over the flow term, we obtain the diffusion equation

$$\frac{1}{\sigma\mu_o} \nabla^2 \underline{B}^* = \frac{\partial \underline{B}^*}{\partial t}$$

By dimensional arguments, the characteristic time for magnetic field inhomogeneities to decay or smooth out is shown to be

$$\tau^* = a^2 \sigma \mu_o$$

For disturbances which occur in times short compared to τ^*, the fluid responds as though it had infinite conductivity. The growth times in this report, and in reference 5, are proportional to $1/c_i\Omega$. For the infinite conductivity results of this report to apply, the growth times should be short compared to τ^*. Thus, the second criterion becomes

$$\tau^*\Omega c_i = \Omega a^2 \sigma \mu_o c_i >> 1$$

While the first criterion required only that the magnetic Reynolds number be large, the second criterion requires that the product of the magnetic Reynolds number and the growth rate must be large. From figure 2, the growth rates c_i are of the order of 3 or less for $m \leq 5$. In applying the second criterion, the growth rate c_i shall be given the value 3.0.

For the plasma and liquid-metal systems of interest, the following table gives typical values for magnetic Reynolds number, magnetic decay time, and the ratio of magnetic decay time to growth time.

	Magnetic Reynolds number, $\Omega a^2 \sigma \mu_o$	Magnetic decay time, τ^*, sec	Ratio of magnetic decay time to growth time, $\tau^*\Omega c_i$
Plasma	0.6	3×10^{-3}	1.8
Sodium	2520	12.6	7560
Potassium	1680	8.4	5040
Mercury	260	1.3	780

For the wheel-flow reactor, both the magnetic Reynolds number and $\tau^* \Omega c_i$ are of order 1, implying that the magnetic field will have very little influence and that the stability of such a system would be better represented by the hydrodynamic analysis of reference 5.

For the liquid metals, the opposite conclusion applies. The magnetic effects clearly dominate. Magnetic stabilization of a free surface may have application to those space power conversion systems that employ a liquid metal as the working fluid in a Rankine power cycle. A stable liquid-vapor interface must be maintained in the condenser to prevent vapor-locking of the condensate pump. Although there is a zero-gravity environment, there are local mechanical disturbances that tend to destabilize the interface. The results of this report, and those found in reference 13, indicate that strong magnetic fields and short wavelengths will provide a stabilizing influence. The disturbances can be restricted to short wavelengths by making the boundaries small. If the confining walls are made of metal, induced voltages in the fluid will be short-circuited. And like a short-circuited homopolar generator, the movement of the fluid would be severely restricted.

SUMMARY OF RESULTS

The stability of an incompressible two-fluid wheel flow to infinitesimal helical disturbances has been considered. The inner fluid is heavy and has infinite electrical conductivity, while the outer fluid is light and is nonconducting. Both fluids are inviscid. A constant and uniform axial magnetic field is impressed on both fluids. Important results are:

1. The growth rates of both propagating and nonpropagating waves are diminished by increasing axial magnetic field.
2. For specific values of axial and azimuthal wave number, a value of magnetic field can be found which reduces the growth rates to zero.
3. Growth rates at a specific axial wave number increase for waves with increasing azimuthal wave number.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, April 23, 1969,
129-02-08-02-22.

APPENDIX A

SYMBOLS

A	area
$\mathscr{A}$	constant in solution of differential equation
a	interface radius
B	magnetic field
$\mathscr{B}$	constant in solution of differential equation
b	disturbance magnetic field
c	wave frequency observed in rotating frame, $\omega + m$
c_i	growth rate
D	defined by eq. (87)
E	electric field
F	defined by eq. (91)
G	defined by eq. (88)
$H_m(x)$	Hankel function of first kind, argument x
$\mathscr{H}_m$	defined by eq. (82)
$I_m(x)$	modified Bessel function of first kind, argument x
Im	imaginary part of
J	current density
$J_m(x)$	Bessel function of first kind, argument x
$\mathscr{J}_m$	defined by eq. (83)
$K_m(x)$	modified Bessel function of second kind, argument x
k	axial wave number
L	length
m	azimuthal wave number
n	vector normal to interface surface
$\mathscr{n}$	first-order perturbation of unit vector normal to surface
P	pressure
p	disturbance pressure
Q	defined by eq. (89) or (92) depending on subscript
$q(r)$	complex disturbance amplitude
R	outer radius
Re	real part of
$\mathscr{R}$	defined by eq. (98)
S	ratio of Alfven speed to reference speed
s	surface variable
t	time
V	velocity
v	disturbance velocity
$W\{a, b\}$	Wronskian of a and b
r, θ, z	cylindrical coordinates
x, y, z	Cartesian coordinates
δ	defined by eq. (90) or (93) depending on subscript
ξ	radial displacement of interface from equilibrium
μ_o	permeability of free space

ρ	density
σ	electrical conductivity
Ω	angular wheel-flow velocity
ω	complex angular disturbance velocity

Subscripts:

c	equilibrium quantity at $r^* = 0$
critical	value of k for which c goes to zero
i	imaginary
re	real
ref	reference quantity
α	dummy script
0	equilibrium quantity
1	inner fluid
2	outer fluid
_	vector

Superscripts:

'	prime denoting differentiation with respect to r
*	dimensional quantity
^	unit vector

APPENDIX B

DERIVATION OF BOUNDARY CONDITIONS

The boundary condition equations (eqs. (15) to (17)) are found by integrating appropriate differential equations across the interface. In the real physical situation, there is a rapid but continuous transition of fluid properties across a thin layer separating the two fluids. Therefore, the differential equations that we use must be general enough to describe the flow on either side of, and through, the transition region. Therefore, we assume that both fluids satisfy the following conditions: the conductivity is finite; viscosity, space charge, and displacement current can be neglected; and the magnetic permeability is that of free space. Any idealized fluid properties that are different on opposite sides of the interface, such as infinite conductivity on one side and zero conductivity on the other, must be introduced into the boundary conditions after the limits are taken.

Following the approach used in reference 14, we form a volume element, fixed in space, which instantaneously includes the interface (fig. 5). The volume will be a right cylinder such that at time t the lower face of the cylinder coincides with the interface. As time increases to $t + \delta t$, the interface moves up to the position shown in figure 5. The length of the cylinder Δ is considered to be very small. Thus, in the limit, the side walls make no contribution to those surface integrals with finite integrands. The surfaces at the top and bottom of the cylinder are small enough in area A to allow a

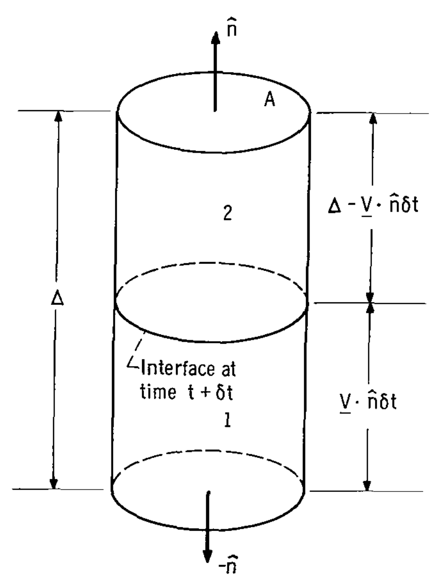


Figure 5. - Volume element intersecting the interface between regions 1 and 2.

surface integral to be approximated as the product of the integrand and the surface area.

The use of a fixed volume in space precludes the difficulties associated with moving reference frames. When integrating across the interface, while in a frame moving with the interface, the field quantities in the differential equations would have to be transformed into those fields that the moving observer would experience. The resulting boundary conditions for the moving frame then have to be rewritten in terms of the fields observed in the laboratory frame.

Derivation of Equation (16)

Equation (16) is obtained by integrating the conservation-of-mass equation over the volume element of figure 5. The integral form of the conservation of mass over a fixed volume is

$$\frac{\partial}{\partial t}\int_v \rho dv + \int_v \nabla \cdot (\rho \underline{V}) dv = 0 \tag{B1}$$

Using Gauss' theorem to transform the second integral of equation (B1) into a surface integral, we get

$$\frac{\partial}{\partial t}\int_v \rho \, dv + \int_s \rho \underline{V} \cdot \hat{n} \, ds = 0 \tag{B2}$$

The first integral in equation (B2) is approximated as

$$\frac{\partial}{\partial t}\int_v \rho \, dv = \frac{1}{\delta t}\left\{\int_v (\rho)_{t+\delta t} \, dv - \int_v (\rho)_t \, dv\right\} \tag{B3}$$

$$\frac{\partial}{\partial t}\int_v \rho \, dv = \frac{1}{\delta t}\left\{\left(\rho_1\right)_{t+\delta t}\underline{V} \cdot \hat{n}\delta t A + \left(\rho_2\right)_{t+\delta t}(\Delta - \underline{V} \cdot \hat{n}\delta t)A - \left(\rho_2\right)_t \Delta A\right\} \tag{B4}$$

Using a Taylor expansion, we get

$$\left(\rho_1\right)_{t+\delta t} = \left(\rho_1\right)_t + \left(\frac{\partial \rho_1}{\partial t}\right)_t \delta t + . . . \tag{B5}$$

$$\left(\rho_2\right)_{t+\delta t} = \left(\rho_2\right)_t + \left(\frac{\partial \rho_2}{\partial t}\right)_t \delta t + \ . \ . \ . \tag{B6}$$

Substitute for $\left(\rho_1\right)_{t+\delta t}$ and $\left(\rho_2\right)_{t+\delta t}$ from equations (B5) and (B6), respectively, into equation (B4) to obtain

$$\frac{\partial}{\partial t}\int_v \rho \, dv = \frac{1}{\delta t}\left\{\left[\left(\rho_1\right)_t + \left(\frac{\partial \rho_1}{\partial t}\right)_t \delta t\right]\underline{V} \cdot \hat{n}\delta t A + \left[\left(\rho_2\right)_t + \left(\frac{\partial \rho_2}{\partial t}\right)_t \delta t\right](\Delta - \underline{V} \cdot \hat{n}\delta t)A - \left(\rho_2\right)_t \Delta A\right\} \tag{B7}$$

$$\frac{\partial}{\partial t}\int_v \rho \, dv = \left\{\left[\left(\rho_1\right)_t + \left(\frac{\partial \rho_1}{\partial t}\right)_t \delta t\right] - \left[\left(\rho_2\right)_t + \left(\frac{\partial \rho_2}{\partial t}\right)_t \delta t\right]\right\}\underline{V} \cdot \hat{n}A + \left(\frac{\partial \rho_2}{\partial t}\right)_t \Delta A \tag{B8}$$

Now passing to the limit we obtain from equation (B8)

$$\lim_{\delta t, \Delta \to 0} \frac{\partial}{\partial t}\int_v \rho \, dv = (\rho_1 - \rho_2)\underline{V} \cdot \hat{n}A \tag{B9}$$

The second integral in equation (B2) can be approximated as

$$\int_s \rho\underline{V} \cdot \hat{n} \, ds = \rho_1\underline{V}_1 \cdot (-\hat{n})A + \rho_2\underline{V}_2 \cdot \hat{n}A + \mathcal{O}(\Delta) \tag{B10}$$

where $\mathcal{O}(\Delta)$ is the contribution from the cylindrical surface. This term will vanish in the limit $\Delta \to 0$ because the integrand is finite over the cylindrical surface. Thus, the limit of equation (B10) becomes

$$\lim_{\Delta \to 0}\int_s \rho\underline{V} \cdot \hat{n} \, ds = (\rho_2\underline{V}_2 \cdot \hat{n} - \rho_1\underline{V}_1 \cdot \hat{n})A \tag{B11}$$

Substituting equations (B9) and (B11) into equation (B2) gives

$$(\rho_1 - \rho_2)\underline{V} \cdot \hat{n} + \rho_2\underline{V}_2 \cdot \hat{n} - \rho_1\underline{V}_1 \cdot \hat{n} = 0$$

And on rearrangement, we have

$$\rho_1(\underline{V} \cdot \hat{n} - \underline{V}_1 \cdot \hat{n}) - \rho_2(\underline{V} \cdot \hat{n} - \underline{V}_2 \cdot \hat{n}) = 0 \tag{B12}$$

Equation (B12) is satisfied if

$$\boxed{\underline{V} \cdot \hat{n} = \underline{V}_1 \cdot \hat{n} = \underline{V}_2 \cdot \hat{n}} \tag{16}$$

Equation (16) is the kinematic condition that the fluid velocity component normal to the boundary must equal the velocity of the boundary normal to itself. This applies to our situation, where the two fluids are immiscible. In this case no mass flows across the surface, and the fluids on opposite sides of the surface remain in contact with the surface.

When equation (B12) is satisfied but equation (16) is not, we have the boundary condition for a shock or detonation front where fluid can cross the interface.

Derivation of Equation (17)

Equation (17) is obtained by integrating the $\nabla \cdot \underline{B} = 0$ equation (eqs. (9) and (14)) over the volume element of figure 5.

$$\int_V \nabla \cdot \underline{B} \, dv = 0 \tag{B13}$$

Using Gauss' theorem to convert to a surface integral, equation (B13) becomes

$$\int_S \underline{B} \cdot \hat{n} \, ds = 0 \tag{B14}$$

The surface integral can be approximated by

$$\int_S \underline{B} \cdot \hat{n} \, ds = \underline{B}_1 \cdot (-\hat{n})A + \underline{B}_2 \cdot \hat{n}A + \mathcal{O}(\Delta) = 0 \tag{B15}$$

And passing to the limit gives

$$\lim_{\Delta \to 0} \int_S \underline{B} \cdot \hat{n}\, ds = \hat{n} \cdot (\underline{B}_2 - \underline{B}_1)A = 0 \tag{B16}$$

or

$$\boxed{\hat{n} \cdot (\underline{B}_2 - \underline{B}_1) = 0} \tag{17}$$

Derivation of Equation (15)

Equation (7) is applicable to both sides of the interface, subject to the conditions noted at the beginning of this appendix. Hence, we integrate equation (7) over the volume element of figure 5.

$$\int_V \rho\left[\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V}\right] dv = \frac{1}{\mu_o}\int_V (\underline{B} \cdot \nabla)\underline{B}\, dv - \int_V \nabla\left(P + \frac{B^2}{2\mu_o}\right) dv \tag{B17}$$

By making use of the continuity equation and the identities given below, we can replace the volume integral on the left side of equation (B17) by one that is more convenient to evaluate. Accordingly, we write the differential form of the continuity equation as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{V}) \tag{B18}$$

We also have the identity

$$\rho \frac{\partial \underline{V}}{\partial t} = \frac{\partial(\rho \underline{V})}{\partial t} - \underline{V} \frac{\partial \rho}{\partial t} \tag{B19}$$

If we substitute $\partial\rho/\partial t$ from equation (B18) into equation (B19), we get

$$\rho \frac{\partial \underline{V}}{\partial t} = \frac{\partial(\rho \underline{V})}{\partial t} + \underline{V} \cdot \nabla(\rho \underline{V}) \tag{B20}$$

Using equation (B20), the integrand of the volume integral on the left side of equation (B17) becomes

$$\rho \frac{\partial \underline{V}}{\partial t} + \rho(\underline{V} \cdot \nabla)\underline{V} = \frac{\partial}{\partial t}(\rho\underline{V}) + \rho(\underline{V} \cdot \nabla)\underline{V} + \underline{V}\,\nabla \cdot (\rho\underline{V}) \tag{B21}$$

or

$$\rho \frac{\partial \underline{V}}{\partial t} + \rho(\underline{V} \cdot \nabla)\underline{V} = \frac{\partial}{\partial t}(\rho\underline{V}) + \nabla \cdot (\rho\underline{V}\,\underline{V}) \tag{B22}$$

Substituting equation (B22) into equation (B17) gives

$$\int_v \frac{\partial}{\partial t}(\rho\underline{V})dv + \int_v \nabla \cdot (\rho\underline{V}\,\underline{V})dv = \frac{1}{\mu_o}\int_v (\underline{B} \cdot \nabla)\underline{B}\ dv - \int_v \nabla\left(P + \frac{B^2}{2\mu_o}\right)dv \tag{B23}$$

By application of the various forms of Gauss' theorem, the last three integrals in equation (B23) can be converted into surface integrals. Thus,

$$\int_v \nabla \cdot (\rho\underline{V}\,\underline{V})dv = \int_s \rho\underline{V}\,\underline{V} \cdot \hat{n}\ ds \tag{B24}$$

$$\int_v (\underline{B} \cdot \nabla)\underline{B}\ dv = \int_s \underline{B}\,\underline{B} \cdot \hat{n}\ ds \tag{B25}$$

$$\int_v \nabla\left(P + \frac{B^2}{2\mu_o}\right)dv = \int_s \left(P + \frac{B^2}{2\mu_o}\right)\hat{n}\ ds \tag{B26}$$

Substituting equations (B24) to (B26) into equation (B23) gives

$$\frac{\partial}{\partial t}\int_v (\rho\underline{V})dv + \int_s \rho\underline{V}\,\underline{V} \cdot \hat{n}\ ds = \frac{1}{\mu_o}\int_s \underline{B}\,\underline{B} \cdot \hat{n}\ ds - \int_s \left(P + \frac{B^2}{2\mu_o}\right)\hat{n}\ ds \tag{B27}$$

Each integral in equation (B27) shall be separately evaluated for the volume element of figure 5.

The first integral in equation (B27) is evaluated in the same manner as for the first integral in equation (B2). Thus, we can write the result by replacing ρ in equation (B9) by $\rho\underline{V}$, and we get

$$\lim_{\delta t,\, \Delta \to 0} \frac{\partial}{\partial t} \int_v \rho\underline{V}\, dv = (\rho_1\underline{V}_1 - \rho_2\underline{V}_2)\underline{V} \cdot \hat{n}A \tag{B28}$$

The second integral in equation (B27) is approximated as

$$\int_s \rho\underline{V}\underline{V} \cdot \hat{n}\, ds = \rho_2\underline{V}_1\underline{V}_1 \cdot (-\hat{n})A + \rho_2\underline{V}_2\underline{V}_2 \cdot \hat{n}A + \mathcal{O}(\Delta) \tag{B29}$$

Passing to the limit gives

$$\lim_{\Delta \to 0} \int_s \rho\underline{V}\underline{V} \cdot \hat{n}\, ds = (\rho_2\underline{V}_2\underline{V}_2 \cdot \hat{n} - \rho_1\underline{V}_1\underline{V}_1 \cdot n)A \tag{B30}$$

We may use equation (16) to reduce equation (B30) to

$$\lim_{\Delta \to 0} \int_s \rho\underline{V}\underline{V} \cdot \hat{n}\, ds = (\rho_2\underline{V}_2 - \rho_1\underline{V}_1)\underline{V} \cdot \hat{n}A \tag{B31}$$

The third integral in equation (B27) is approximated as

$$\int_s \underline{B}\underline{B} \cdot \hat{n}\, ds = \underline{B}_1\underline{B}_1 \cdot (-\hat{n})A + \underline{B}_2\underline{B}_2 \cdot \hat{n}A + \mathcal{O}(\Delta) \tag{B32}$$

Passing to the limit gives

$$\lim_{\Delta \to 0} \int_s \underline{B}\underline{B} \cdot \hat{n}\, ds = (\underline{B}_2\underline{B}_2 \cdot \hat{n} - \underline{B}_1\underline{B}_1 \cdot \hat{n})A \tag{B33}$$

By using equation (17), we can simplify equation (B33) to

$$\lim_{\Delta \to 0} \int_s \underline{B}\underline{B} \cdot \hat{n}\, ds = (\underline{B}_2 - \underline{B}_1)\underline{B} \cdot \hat{n}A \tag{B34}$$

where

$$\underline{B} \cdot \hat{n} = \underline{B}_1 \cdot \hat{n} = \underline{B}_2 \cdot \hat{n} \tag{B35}$$

Finally, the fourth integral in equation (B27) is approximated as

$$\int_S \left(P + \frac{B^2}{2\mu_o}\right)\hat{n}\, ds = \left[\left(P_1 + \frac{B_1^2}{2\mu_o}\right)(-\hat{n}) + \left(P_2 + \frac{B_2^2}{2\mu_o}\right)\hat{n}\right]A + \mathcal{O}(\Delta) \tag{B36}$$

Passing to the limit gives

$$\lim_{\Delta \to 0} \int_S \left(P + \frac{B^2}{2\mu_o}\right)\hat{n}\, ds = \hat{n}\left[\left(P_2 + \frac{B_2^2}{2\mu_o}\right) - \left(P_1 + \frac{B_1^2}{2\mu_o}\right)\right]A \tag{B37}$$

Now substitute equations (B28), (B31), (B34), and (B37) into equation (B27) to obtain

$$(\rho_1 \underline{V}_1 - \rho_2 \underline{V}_2)\underline{V} \cdot \hat{n} + (\rho_2 \underline{V}_2 - \rho_1 \underline{V}_1)\underline{V} \cdot \hat{n} = (\underline{B}_2 - \underline{B}_1)\underline{B} \cdot \hat{n} - \hat{n}\left[\left(P_2 + \frac{B_2^2}{2\mu_o}\right) - \left(P_1 + \frac{B_1^2}{2\mu_o}\right)\right] \tag{B38}$$

The two terms on the left side of equation (B38) cancel each other, while the two terms on the right side are orthogonal to each other. The orthogonality condition follows from the fact that the vector $\underline{B}_2 - \underline{B}_1$ lies in the surface of the interface. This is easily seen by noting that the magnetic field vectors $\underline{B}_1$ and $\underline{B}_2$ may be resolved into components normal and tangential to the interface, $\underline{B}_n$ and $\underline{B}_t$ respectively. Thus,

$$\underline{B}_2 - \underline{B}_1 = \underline{B}_{n2} + \underline{B}_{t2} - \underline{B}_{n1} - \underline{B}_{t1} \tag{B39}$$

But from equation (17), the equation of continuity of the normal component of magnetic field at the interface, we have that

$$\underline{B}_{n2} = \underline{B}_{n1} \tag{B40}$$

Thus,

$$\underline{B}_2 - \underline{B}_1 = \underline{B}_{t2} - \underline{B}_{t1} \tag{B41}$$

Therefore, each of the two terms on the right side of equation (B38) must independently be zero. This results in the following two boundary conditions:

$$\boxed{\hat{n}\left[\left(P_2 + \frac{B_2^2}{2\mu_o}\right) - \left(P_1 + \frac{B_1^2}{2\mu_o}\right)\right] = 0} \tag{15}$$

$$\boxed{(\underline{B}_2 - \underline{B}_1)(\underline{B} \cdot \hat{n}) = 0} \tag{B42}$$

Equation (B42) is not used in the text because it does not yield any zero-order or first-order information for the problem treated in this report. However, equation (B42) can be interpreted in general. It is interesting to note that if $\underline{B}_2 - \underline{B}_1 \neq 0$ (i.e., when there are surface currents, which in turn means that one of the fluids is a perfect conductor) equation (B42) requires that

$$\underline{B} \cdot \hat{n} = \underline{B}_1 \cdot \hat{n} = \underline{B}_2 \cdot \hat{n} = 0 \tag{B43}$$

We infer from equation (B43) that the field on both sides of the interface is wholly tangential to the interface. This follows physically from the infinite conductivity assumption which causes the field lines and the fluid to be "frozen" together. We know from equation (18) that a particle, once on the surface, always remains on the surface. Since the magnetic field lines and particles are always "frozen" together, we should expect that the magnetic field lines, once on the surface, will remain always on the surface.

If $\underline{B}_{02} \neq \underline{B}_{01}$, equation (B42) yields first-order information about the boundary conditions. To demonstrate this we write out the zero- and first-order terms in equation (B42).

$$(\underline{B}_2 - \underline{B}_1)(\underline{B} \cdot \hat{n}) = 0 \tag{B44}$$

$$(\underline{B}_{02} + \underline{b}_2 - \underline{B}_{01} - \underline{b}_1)\left[(\underline{B}_{0\alpha} + \underline{b}_\alpha) \cdot (\hat{r} + \underline{\boldsymbol{n}})\right] = 0 \tag{B45}$$

where $\alpha = 1, 2$

$$(\underline{B}_{02} + \underline{b}_2 - \underline{B}_{01} - \underline{b}_1)(\underline{B}_{0\alpha} \cdot \hat{r} + \underline{B}_{0\alpha} \cdot \underline{\boldsymbol{n}} + \underline{b}_\alpha \cdot \hat{r} + \underline{b}_\alpha \cdot \underline{\boldsymbol{n}}) = 0 \tag{B46}$$

The zero-order terms of this equation are

$$(\underline{B}_{02} - \underline{B}_{01})(\underline{B}_{0\alpha} \cdot \hat{r}) = 0 \tag{B47}$$

The first-order terms are

$$(\underline{B}_{02} - \underline{B}_{01})(\underline{B}_{0\alpha} \cdot \underline{\boldsymbol{n}} + \underline{b}_{\alpha} \cdot \hat{r}) + (\underline{b}_{2} - \underline{b}_{1})(\underline{B}_{0\alpha} \cdot \hat{r}) = 0 \tag{B48}$$

If $\underline{B}_{02} \neq \underline{B}_{01}$, $\underline{B}_{0\alpha} \cdot \hat{r} = 0$ from equation (B47), and equation (B48) reduces to

$$(\underline{B}_{02} - \underline{B}_{01})(\underline{B}_{0\alpha} \cdot \underline{\boldsymbol{n}} + \underline{b}_{\alpha} \cdot \hat{r}) = 0 \tag{B49}$$

and since $\underline{B}_{02} \neq \underline{B}_{01}$, equation (B49) requires that

$$\underline{B}_{0\alpha} \cdot \underline{\boldsymbol{n}} + \underline{b}_{\alpha} \cdot \hat{r} = 0 \qquad \alpha = 1, 2 \tag{B50}$$

Equation (B50) is the first-order boundary condition that results when $\underline{B}_{02} \neq \underline{B}_{01}$. In the problem considered in this report, $\underline{B}_{02} = \underline{B}_{01}$, and thus equation (B50) does not apply. Furthermore, in this report $\underline{B}_{0\alpha} \cdot \hat{r} = 0$; hence, equation (B48) yields no first-order information.

Other Boundary Conditions

In reference 6, Wilhelm presents a boundary condition (eq. 1.7 of ref. 6) that was derived by integrating equation (10) of this report across the interface. Now equation (10) was obtained by substituting equation (3), which holds only for infinite conductivity, into equation (6). Since equation (10) does not hold on both sides of the interface, it cannot be used to obtain boundary conditions. Fortunately, equation 4.6 of reference 6 can be obtained from equation (B38), so Wilhelm's results are valid.

In summary, the following table shows the differential equations and the corresponding boundary conditions obtained by integrating the equations across the interface.

Differential equation	Boundary condition
$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$	$\hat{n} \cdot (\underline{V}_2 - \underline{V}_1) = 0$
$\nabla \cdot \underline{B} = 0$	$\hat{n} \cdot (\underline{B}_2 - \underline{B}_1) = 0$
$\rho\left[\frac{\partial \underline{V}}{\partial t} + (\underline{V} \cdot \nabla)\underline{V}\right] = \frac{1}{\mu_o} (\underline{B} \cdot \nabla)\underline{B} - \nabla \left(P + \frac{B^2}{2\mu_o}\right)$	$\hat{n}\left[\left(P_2 + \frac{B_2^2}{2\mu_o}\right) - \left(P_1 + \frac{B_1^2}{2\mu_o}\right)\right] = 0$
	$(\underline{B}_2 - \underline{B}_1)(\underline{B} \cdot \hat{n}) = 0$

APPENDIX C

LIMITING SOLUTION OF DISPERSION RELATION FOR SMALL AXIAL WAVE NUMBER

The dispersion relation equation (42) is to be solved in the limit of small axial wave number k. The Bessel functions become

$$\frac{\left[\frac{d}{dr} J_m(ik\delta_1 r)\right]_{r=1}}{J_m(ik\delta_1)} = m + \ldots \tag{C1}$$

$$\frac{\left[\frac{d}{dr} H_m(ik\delta_2 r)\right]_{r=1}}{H_m(ik\delta_2 r)} = -m + \ldots \tag{C2}$$

$$\frac{\left[\frac{d}{dr} H_m(ikr)\right]_{r=1}}{H_m(ik)} = -m + \ldots \tag{C3}$$

For $k \rightarrow 0$

$$\delta_1^2 \rightarrow \delta_2^2 = 1 - \frac{4}{c^2} \tag{C4}$$

$$Q_1 \rightarrow Q_2 = ic \tag{C5}$$

The dispersion relation can be rewritten

$$(c^2 - 4)(\mathscr{R} c^2 - 2c + m) = 0 \tag{C6}$$

where

$$\mathscr{R} = \frac{\frac{\rho_1}{\rho_2} + 1}{\frac{\rho_1}{\rho_2} - 1} \tag{C7}$$

The solutions to the dispersion relation are

$$c = \pm 2 \tag{C8}$$

$$c = \frac{1}{\mathscr{R}}\left(1 \pm i\sqrt{\mathscr{R}m - 1}\right) \tag{C9}$$

For the solutions given in equation (C8), $\delta_1^2 = \delta_2^2 \cong 0$. These solutions are rejected because the condition that the real part of $k\delta_2$ is greater than zero does not hold.

The solutions in equation (C9) are both valid solutions. The solution with the negative imaginary root grows while the solution with the positive imaginary root decays. Although both are possible, the growth solution is of more interest here since it persists.

APPENDIX D

DETERMINATION OF THE POINT WHERE WAVE FREQUENCY $c = 0$

Let $c = 0$. The dispersion relation (eq. (86)) becomes

$$0 = \frac{\rho_2}{\rho_1} - 1 + k^2 B_0^2 \left[\frac{J_m(ik)}{J_m'(ik)} - \frac{H_m(ik)}{H_m'(ik)} \right] \tag{D1}$$

Differentiating the Bessel functions and rewriting them as modified Bessel functions gives

$$1 - \frac{\rho_2}{\rho_1} = k^2 B_0^2 \left[\frac{I_m(k)}{kI_{m-1}(k) - mI_m(k)} + \frac{K_m(k)}{mK_m(k) - kK_{m-1}(k)} \right] \tag{D2}$$

$$1 - \frac{\rho_2}{\rho_1} = k^2 B_0^2 \left\{ \frac{kI_m(k)K_{m-1}(k) + kI_{m-1}(k)K_m(k)}{k^2 I_{m-1}(k)K_{m-1}(k) + mk\left[I_{m-1}(k)K_m(k) - I_m(k)K_{m-1}(k)\right] - m^2 I_m(k)K_m(k)} \right\} \tag{D3}$$

From the Wronskian of the modified Bessel functions

$$W\left[K_{m-1}(k), I_{m-1}(k)\right] = I_{m-1}(k)K_m(k) + I_m(k)K_{m-1}(k) = \frac{1}{k} \tag{D4}$$

Thus, the numerator of the factor containing Bessel functions equals 1. Rearranging terms gives

$$\frac{k^2 B_0^2}{1 - \frac{\rho_2}{\rho_1}} = k^2 I_{m-1}(k)K_{m-1}(k) + mk\left[I_{m-1}(k)K_m(k) - I_m(k)K_{m-1}(k)\right] - m^2 I_m(k)K_m(k) \tag{D5}$$

Solving equation (D5) for B_z yields the value of k at which $c = 0$. For several limiting cases, the equation can be further simplified. For all m as $k \rightarrow \infty$, equation (D5) reduces to

$$\frac{B_0^2}{1 - \frac{\rho_2}{\rho_1}} = \frac{1}{2k} \tag{D6}$$

For $m = 0$, $k \to 0$, equation (D5) becomes

$$\frac{B_0^2}{1 - \frac{\rho_2}{\rho_1}} = \frac{1}{2} \tag{D7}$$

For $m = 1$, $k \to 0$, equation (D5) becomes

$$\frac{B_0^2}{1 - \frac{\rho_2}{\rho_1}} = \frac{1}{2k^2} - \frac{1}{2} \ln k \tag{D8}$$

For $m > 1$, $k \to 0$, equation (D5) becomes

$$\frac{B_0^2}{1 - \frac{\rho_2}{\rho_1}} = \frac{1}{4(m - 1)} + \frac{m}{2k^2} \tag{D9}$$

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